Interplay between quark-antiquark and diquark condensates in vacuum in a two-flavor Nambu-Jona-Lasinio model*

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By means of a relativistic effective potential, we have analytically researched competition between the quark-antiquark condensates $\langle \bar{q}q \rangle$ and the diquark condensates $\langle qq \rangle$ in vacuum in ground state of a two-flavor Nambu-Jona-Lasinio (NJL) model and obtained the $G_S - H_S$ phase diagram, where G_S and H_S are the respective four-fermion coupling constants in scalar quark-antiquark channel and scalar color anti-triplet diquark channel. The results show that, in the chiral limit, there is only the pure $\langle \bar{q}q \rangle$ phase when $G_S/H_S > 2/3$, and as G_S/H_S decreases to $2/3 > G_S/H_S \geq 0$ one will first have a coexistence phase of the condensates $\langle \bar{q}q \rangle$ and $\langle qq \rangle$ and then a pure $\langle qq \rangle$ phase. In non-zero bare quark mass case, the critical value of G_S/H_S at which the pure $\langle \bar{q}q \rangle$ phase will transfer to the coexistence phase of the condensates $\langle \bar{q}q \rangle$ and $\langle qq \rangle$ will be less than 2/3. Our theoretical results, combined with present phenomenological fact that there is no diquark condensates in the vacuum of QCD, will also impose a real restriction to any given two-flavor NJL model which is intended to simulate QCD, i.e. in such model the resulting smallest ratio G_S/H_S after the Fierz transformations in the Hartree approximation must be larger than 2/3. A few phenomenological QCD-like NJL models are checked and analyzed.

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I. INTRODUCTION

It is well known that in Quantum Chromodynamics (QCD) at temperature $T = 0$ and quark chemical potential $\mu = 0$, i.e. in vacuum, quarks are in chiral symmetry breaking phase owing to the quark-antiquark condensates $\langle \bar{q}q \rangle \neq 0$ [1, 2, 3, 4, 5, 6, 7]. Recently It has also been clear that, at low temperature and high baryonic density, the quarks will be in color superconducting phase owing to some diquark condensates being nonzeroes [8, 9, 10, 11]. Research on diquark condensates and color superconducting has also been extended to the region of low temperature and moderate quark chemical potential where perturbative or present-day lattice calculations are inaccessible, by means of different phenomenological models including the Nambu-Jona-Lasinio (NJL)-type models [2, 4, 12]. Since the interactions between two quarks in the antitriplet of the color $SU_c(3)$ gauge group are attractive, one can reasonably inquire if the diquark condensates could be generated in vacuum. This problem has been researched or touched on by means of some phenomenological models [13, 14, 15]. Some results from a few given NJL-type models were obtained. However, it seems that, for a thorough understanding of the diquark condensates in a NJL model in vacuum, especially of the interplay between the diquark and the quark-antiquark condensates in vacuum, a general and systematic analysis is still necessary. In addition, such research itself has certainly theoretical interest if one treats the NJL model as a useful field theory model. The NJL model as a phenomenological description of QCD, is usually used in discussion of chiral symmetry breaking but very little in the diquark condensation problem in vacuum. Then what is the theoretical reason of this situation? In fact, in any NJL model, owing to the Fierz transformation [12], one is always allowed to include both the interactions of $(\bar{q}q)^2$-form and $(qq)^2$-form which could lead to generation of the quark-antiquark and the diquark condensates respectively. As long as applying a NJL model to diquark condensation problem in vacuum, one will inevitably face to mutual competition between the diquark condensates and the quark-antiquark condensates. A serious research on such mutual competition in a NJL model will lead to some condition in which the diquark condensates could be generated, or say, removed in vacuum. By this condition, combined with the phenomenological requirements of QCD in vacuum, one will be able to answer why in conventional NJL model description of QCD in vacuum one can completely neglect existence of the diquark condensates. In fact, the condition to remove the diquark condensates in vacuum will become a useful restriction to any "realistic" NJL model.

Motivated by the above ideas, in this paper we will make a general analysis of the mutual competition between the quark-antiquark and diquark condensates in a NJL-type model in vacuum. The paper is arranged as follows. In Sect.II we will present the model and derive its effective potential, in Sect.III determine the ground states in different conditions and in Sect.IV come to our conclusions and discussions.

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II. MODEL AND EFFECTIVE POTENTIAL

we start with a two-flavor NJL model which could be used to simulate QCD and derive its effective potential containing the order parameters respectively corresponding to quark-antiquark and diquark condensates in mean field approximation. As is well known, in the mean field approximation, there is so called Fierz ambiguity [11, 16]. However, throughout the whole discussions in this paper, we will follow Ref.[12] and use the Fierz-transformed four-fermion couplings in the Hartree approximation to avoid double counting. To simplify the problem and to make it possible to conduct the demonstration in a completely analytic way, we will first omit the small bare masses of the u and d quarks (i.e. take the chiral limit). Thus the Lagrangian of the two-flavor NJL model may be written in the well-known form by

$$\begin{aligned}\mathcal{L} &= \bar{q}i\,\partial\!\!\!/ q + G_S[(\bar{q}q)^2 + (\bar{q}i\gamma_5\vec{\tau}q)^2] \\ &\quad + H_S \sum_{A=2,5,7} (\bar{q}i\gamma_5\tau_2\lambda_A q^C)(\bar{q}^C i\gamma_5\tau_2\lambda_A q), \qquad (1)\end{aligned}$$

with the quark fields q in the $SU_f(2)$ doublet and the $SU_c(3)$ triplets, i.e.

$$q = \begin{pmatrix} u_i \\ d_i \end{pmatrix} \quad i = r, g, b,$$

where the subscripts $i = r, g, b$ denote the three colors (red, green and blue) of quarks. In Eq.(1) we have used the denotations $\partial\!\!\!/ \equiv \gamma^\mu\partial_\mu$, Pauli matrices $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ acting in the two-flavor space, the Gell-Mann matrices λ_A acting in the three-color space and the charge conjugations q^C and $\bar{q}^C$ of the quark fields. The Lagrangian (1) is $SU_c(3)\otimes SU_{fL}(2)\otimes SU_{fR}(2)\otimes U_f(1)$-invariant. For the sake of making a general discussion of mutual competition between the diquark and the quark-antiquark condensates in the model, we temporarily view the positive coupling constants G_S and H_S as independent and changeable parameters. Assume that the four-fermion interactions can lead to the scalar quark-antiquark condensates

$$\langle\bar{q}q\rangle = \phi \qquad (2)$$

which will break the chiral symmetries $SU_{fL}(2) \otimes SU_{fR}(2)$ down to $SU_{fV}(2)$ and induce three massless pions as corresponding Goldstone bosons. We also suppose that the four-fermion interactions could generate the scalar color-antitriplet diquark and color-triplet diantiquark condensates (after a global $SU_c(3)$ transformation)

$$\langle\bar{q}^C i\gamma_5\tau_2\lambda_2 q\rangle = \delta,\ \langle\bar{q}i\gamma_5\tau_2\lambda_2 q^C\rangle = \delta^*. \qquad (3)$$

It is indicated that in view of the structure of the matrix λ_2 in δ and δ^* in Eq.(3), only the red and green quarks participate in the diquark condensates and the blue quarks do not enter them [12]. For acquiring a qualitative understanding of the diquark condensates in vacuum, we can assume, similar to the BCS ansatz in superconducting theory, physical vacuum as a coherent state of red and green up and down quarks with zero total momentum [9] which contains both quark-quark paring and antiquark-antiquark paring. It is indicated that the NJL model in vacuum i.e. at $T = \mu = 0$ is a relativistic quantum field theory. Hence, despite of absence of net quarks in vacuum, in such theory it is possible that the quark-quark condensates and the antiquark-antiquark condensates are generated simultaneously in vacuum, such as shown by δ and δ^* in Eq.(3). This case is also similar to generation of quark-antiquark condensates in vacuum, where physical vacuum can be assumed to be a coherent state composed of quark-antiquark pairs with zero total momentum[2, 9]. As in the two-flavor color superconducting theory, the condensates (3) will leave the chiral symmetries $SU_{fL}(2) \otimes SU_{fR}(2)$, a "rotated" electric charge $U_{\tilde{Q}}(1)$ and a "rotated" quark number $U'_q(1)$ symmetry unbroken, but will break the $SU_c(3)$ symmetry down to a $SU_c(2)$ symmetry [12]. As a result, if the $SU_c(3)$ group is gauged, then five of the eight gluons will receive a mass through the Higgs mechanism, and if the $SU_c(3)$ group is global, then five massless diquark excitations, which could be some combinations of quark-quark pairs and antiquark-antiquark pairs, will arise as the Goldstone bosons of spontaneous breaking of $SU_c(3)$. We note that the flavor chiral symmetry breaking induced by the quark-antiquark condensates and the color symmetry breaking induced by the diquark condensates are independent each other, so even if the two condensates coexist in ground state, the total physical effects will be only a simple adding up of the effects from the two-forms of symmetry breaking. However, in this paper, we will omit further discussions of the possible physical consequences of the diquark condensates in vacuum, and pay our main attention to the conditions to generate the two-forms of condensates and mutual competition between them. Now define that

$$\sigma = -2G_S\phi,\ \ \Delta = -2H_S\delta,\ \ \Delta^* = -2H_S\delta^* \qquad (4)$$

then in mean-field (Hartree) approximation [12, 17] and in the Nambu-Gorkov (NG) basis [18] with the denotations

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} q \\ q^C \end{pmatrix},\ \ \bar{\Psi} = \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{q}, & \bar{q}^C \end{pmatrix},$$

the Lagrangian (1) may be rewritten by

$$\mathcal{L} = \bar{\Psi}S^{-1}(x)\Psi - \sigma^2/4G_S - |\Delta|^2/4H_S. \qquad (5)$$

In momentum space, the inverse propagator $S^{-1}(x)$ has the expression

$$S^{-1}(p) = \begin{pmatrix} p\!\!\!/ - \sigma & -i\gamma_5\tau_2\lambda_2\Delta \\ -i\gamma_5\tau_2\lambda_2\Delta^* & p\!\!\!/ - \sigma \end{pmatrix}. \qquad (6)$$

Different from the thermodynamic potential at finite temperature and finite chemical potential case, the effective potential in vacuum corresponding to the Lagrangian (5) will be relativistic-invariant. It can be obtained by a direct generalization of the conventional one order-parameter effective potential formula [7, 19] to the case with two order-parameters and in the NG basis. It can be expressed by

$$V(\sigma,\Delta)=\frac{\sigma^2}{4G_S}+\frac{|\Delta|^2}{4H_S}+i\int\frac{d^4p}{(2\pi)^4}\frac{1}{2}\mathrm{Tr}\ln S^{-1}(p)S_0(p) \tag{7}$$

where $S_0(p)$ represents the propagator for massless quarks in the NG basis and the Tr is to be taken over flavor, color, Nambu-Gorkov and Dirac spin degrees of freedom. The factor 1/2 is due to use of the NG basis. Since the blue quarks do not enter the diquark condensates, we will have

$$\begin{aligned}&\mathrm{Tr}\ln S^{-1}(p)S_0(p)=\ln\mathrm{Det}S^{-1}(p)S_0(p)\\&=\ln\mathrm{Det}S^{-1}(p)S_0(p)|_{(r,g)}+\ln\mathrm{Det}S^{-1}(p)S_0(p)|_b\\&=\mathrm{Tr}\ln S^{-1}(p)S_0(p)|_{(r,g)}+\mathrm{Tr}\ln S^{-1}(p)S_0(p)|_b,\end{aligned} \tag{8}$$

where subscripts (r,g) and b mean that the corresponding matrices are limited to (red, green) and blue degrees of freedom in color space respectively. By using the mathematical formula

$$\mathrm{Det}\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\mathrm{Det}[-CB+CAC^{-1}D],$$

where in the above 2×2 matrix, A, B, C and D can be operators (matrices), we obtain

$$\mathrm{Tr}\ln S^{-1}(p)S_0(p)|_{(r,g)} = \ln\mathrm{Det}\frac{\not p}{p^2+i\varepsilon}\begin{pmatrix}\not p-\sigma & -i\gamma_5\tau_2\tau_2^c\Delta\\ -i\gamma_5\tau_2\tau_2^c\Delta^* & \not p-\sigma\end{pmatrix}=16\ln\frac{p^2-\sigma^2-|\Delta|^2+i\varepsilon}{p^2+i\varepsilon}, \tag{9}$$

$$\mathrm{Tr}\ln S^{-1}(p)S_0(p)|_b = \ln\mathrm{Det}\frac{\not p}{p^2+i\varepsilon}\begin{pmatrix}\not p-\sigma & 0\\ 0 & \not p-\sigma\end{pmatrix}\otimes(\mathbf{1}_2^f)=8\ln\frac{p^2-\sigma^2+i\varepsilon}{p^2+i\varepsilon}, \tag{10}$$

where $\tau_2^c=\tau_2$ but it is now acting in the (r,g) two-color space and $(\mathbf{1}_2^f)$ represents the 2×2 unit matrix in the two-flavor space. Substituting Eqs.(9) and (10) into Eq.(8), we may transform Eq.(7) to

$$V(\sigma,\Delta)=\frac{\sigma^2}{4G_S}+\frac{|\Delta|^2}{4H_S}+8i\int\frac{d^4p}{(2\pi)^4}\ln\frac{p^2-\sigma^2-|\Delta|^2+i\varepsilon}{p^2+i\varepsilon}+4i\int\frac{d^4p}{(2\pi)^4}\ln\frac{p^2-\sigma^2+i\varepsilon}{p^2+i\varepsilon} \tag{11}$$

By using the formula

$$I(a^2)=i\int\frac{d^4p}{(2\pi)^4}\ln\frac{p^2-a^2+i\varepsilon}{p^2+i\varepsilon}=\int_0^{a^2}du^2\frac{dI(u^2)}{du^2}$$

it is easy to complete the integrations and obtain the final explicit expression of the effective potential

$$V(\sigma,|\Delta|)=\frac{\sigma^2}{4G_S}+\frac{|\Delta|^2}{4H_S}-\frac{1}{4\pi^2}\left[(3\sigma^2+2|\Delta|^2)\Lambda^2-\frac{\sigma^4}{2}\left(\ln\frac{\Lambda^2}{\sigma^2}+\frac{1}{2}\right)-(\sigma^2+|\Delta|^2)^2\left(\ln\frac{\Lambda^2}{\sigma^2+|\Delta|^2}+\frac{1}{2}\right)\right], \tag{12}$$

where we have made the Wick rotation of the integration variable p^0 and introduced the 4D Euclidean squared momentum cutoff Λ^2 which is assumed to satisfy the conditions $\Lambda^2\gg\sigma^2$ and $\Lambda^2\gg\sigma^2+|\Delta|^2$. It should be indicated that because the 4D Euclidean squared momentum cutoff is used, the effective potential expressed by Eq.(12) will be Lorentz-invariant and this feature is essential for description of the diquark condensates in vacuum. As a comparison, we note that a similar expression to Eq.(12) can be obtained by taking the limit $T=\mu=0$ in some corresponding thermodynamical potential but a 3D momentum cutoff is usually used there. Such a limit expression of the thermodynamical potential, rigorously speaking, is not suitable to the problem of the diquark condensates in vacuum. Since the 3D momentum cutoff breaks the Lorentz invariance, the relevant discussions will be theoretically inconsistent to generating mechanism of the diquark condensates in vacuum which must be built on a relativistic basis, as stated above. Hence only the effective potential (12) with the 4D momentum cutoff will give a rigorous and consistent relativistic description of the problem involving the diquark condensates in vacuum.

III. GROUND STATES

The relativistic effective potential given by Eq.(12) contains two order parameters σ and $|\Delta|$. A obvious advantage of the expression (12) is that one could analytically find out the minimums of $V(\sigma,|\Delta|)$ and then determine the ground state of the system. Simultaneously, one will be able to examine the mutual competition between σ and $|\Delta|$ when the values of the coupling constants G_S and H_S are changed. The extreme value conditions $\partial V(\sigma,|\Delta|)/\partial\sigma=0$ and $\partial V(\sigma,|\Delta|)/\partial|\Delta|=0$ will separately lead to the equations

$$\sigma\left(\frac{1}{2G_S}-\frac{3\Lambda^2}{2\pi^2}+\frac{\sigma^2}{2\pi^2}\ln\frac{\Lambda^2}{\sigma^2}+\frac{\sigma^2+|\Delta|^2}{\pi^2}\ln\frac{\Lambda^2}{\sigma^2+|\Delta|^2}\right)=0 \tag{13}$$

and

$$|\Delta|\left(\frac{1}{2H_S}-\frac{\Lambda^2}{\pi^2}+\frac{\sigma^2+|\Delta|^2}{\pi^2}\ln\frac{\Lambda^2}{\sigma^2+|\Delta|^2}\right)=0. \tag{14}$$

It is easy to see that Eqs.(13) and (14) may have nonzero σ and $|\Delta|$ solutions if

$$G_S\Lambda^2>\pi^2/3 \tag{15}$$

and

$$H_S\Lambda^2>\pi^2/2. \tag{16}$$

The two conditions could be satisfied when either the momentum cutoff Λ is large enough for given coupling constants G_S and H_S or G_S and H_S are large enough for a given Λ. From the second order derivatives of $V(\sigma,|\Delta|)$ denoted by

$$\begin{aligned}
A &\equiv \frac{\partial^2V}{\partial\sigma^2}=\frac{1}{2G_S}-\frac{3\Lambda^2}{2\pi^2}+\frac{3\sigma^2}{2\pi^2}\left(\ln\frac{\Lambda^2}{\sigma^2}-2\right)+\frac{3\sigma^2+|\Delta|^2}{\pi^2}\ln\frac{\Lambda^2}{\sigma^2+|\Delta|^2},\\
B &\equiv \frac{\partial^2V}{\partial\sigma\partial|\Delta|}=\frac{\partial^2V}{\partial|\Delta|\partial\sigma}=2\frac{\sigma|\Delta|}{\pi^2}\left(\ln\frac{\Lambda^2}{\sigma^2+|\Delta|^2}-1\right),\\
C &\equiv \frac{\partial^2V}{\partial|\Delta|^2}=\frac{1}{2H_S}-\frac{\Lambda^2+2|\Delta|^2}{\pi^2}+\frac{\sigma^2+3|\Delta|^2}{\pi^2}\ln\frac{\Lambda^2}{\sigma^2+|\Delta|^2},
\end{aligned} \tag{17}$$

we may define the expression

$$K\equiv\begin{vmatrix}A & B\\ B & C\end{vmatrix}=AC-B^2. \tag{18}$$

The equations (13) and (14) have four different solutions which will be discussed successively as follows.

1) $(\sigma,|\Delta|)=(0,0)$. It is a maximum point of $V(\sigma,|\Delta|)$, since in this case we have

$$A=1/2G_S-3\Lambda^2/2\pi^2<0,$$

$$K=AC=A(1/2H_S-\Lambda^2/\pi^2)>0,$$

if considering the necessary conditions (15) and (16) for existence of non-zero σ and $|\Delta|$ solutions.

2) $(\sigma,|\Delta|)=(\sigma_1,0)$, where $\sigma_1\neq0$ and obeys the equation

$$1/2G_S-3\Lambda^2/2\pi^2+3\sigma_1^2\ln(\Lambda^2/\sigma_1^2)/2\pi^2=0. \tag{19}$$

It may be obtained by means of Eq.(19) that when Λ is large, we have

$$A=3\sigma_1^2[\ln(\Lambda^2/\sigma_1^2)-1]/\pi^2>0$$

and

$$K=A\left(\frac{1}{2H_S}-\frac{1}{3G_S}\right)\begin{cases}>0 & \text{if}\quad G_S/H_S>2/3\\ \leq 0 & \text{if}\quad G_S/H_S\leq 2/3.\end{cases}$$

They indicate that $(\sigma_1,0)$ will be a minimum point of $V(\sigma,|\Delta|)$ when $G_S/H_S>2/3$, however, it will not be an extreme value point of $V(\sigma,|\Delta|)$ when $G_S/H_S\leq2/3$.

3) $(\sigma,|\Delta|)=(0,\Delta_1)$, where $\Delta_1\neq0$ and obeys the equation

$$1/2H_S-\Lambda^2/\pi^2+\Delta_1^2\ln(\Lambda^2/\Delta_1^2)/\pi^2=0. \tag{20}$$

In this case we obtain that

$$A=1/2G_S-3/4H_S-(\Lambda^2/\pi^2-1/2H_S)/2,$$

$$K=A\cdot2\Delta_1^2[\ln(\Lambda^2/\Delta_1^2)-1]/\pi^2.$$

Obviously, the sign of K depends on A's one. We will have $A>0$, if

$$G_S/H_S<1/(1+H_S\Lambda^2/\pi^2). \tag{21}$$

We note that

$$1/(1+H_S\Lambda^2/\pi^2)<2/3, \tag{22}$$

owing to Eq.(16). Equation(21) certainly includes the case of $G_S=0$. Thus $(\sigma,|\Delta|)=(0,\Delta_1)$ will be a minimum point of $V(\sigma,|\Delta|)$ only if Eq.(21) is satisfied, i.e. either the scalar quark-antiquark interactions do not exist or they are very weak, or in other words, the scalar diquark interactions are strong enough opposite to the scalar quark-antiquark interactions.

4)$(\sigma,|\Delta|)=(\sigma_2,\Delta_2)$. The none-zero σ_2 and Δ_2 obey the equations

$$\frac{1}{2G_S}-\frac{3\Lambda^2}{2\pi^2}+\frac{\sigma_2^2}{2\pi^2}\ln\frac{\Lambda^2}{\sigma_2^2}+\frac{\sigma_2^2+\Delta_2^2}{\pi^2}\ln\frac{\Lambda^2}{\sigma_2^2+\Delta_2^2}=0, \quad (23)$$

$$\frac{1}{2H_S}-\frac{\Lambda^2}{\pi^2}+\frac{\sigma_2^2+\Delta_2^2}{\pi^2}\ln\frac{\Lambda^2}{\sigma_2^2+\Delta_2^2}=0. \quad (24)$$

Now from the calculated results by using Eqs.(23) and (24) that

$$A=\frac{\sigma_2^2}{\pi^2}\left[3\left(\ln\frac{\Lambda^2}{\sigma_2^2+\Delta_2^2}-1\right)+\ln\frac{\sigma_2^2+\Delta_2^2}{\sigma_2^2}\right]>0,$$

$$K=\frac{2\sigma_2^2\Delta_2^2}{\pi^4}\left(\ln\frac{\Lambda^2}{\sigma_2^2+\Delta_2^2}-1\right)\left(\ln\frac{\Lambda^2}{\sigma_2^2}-1\right)>0,$$

we may deduce that so long as a non-zero solution (σ_2,Δ_2) exists, it will be a minimum point of $V(\sigma,|\Delta|)$. The remaining problem is to find out the condition in which σ_2 and Δ_2 obeying Eqs.(23) and (24) could simultaneously be equal to non-zero. For this purpose, we note the fact that the function

$$f(a^2)=a^2\ln\frac{\Lambda^2}{a^2}$$

has the feature that

$$\frac{df(a^2)}{da^2}=\ln\frac{\Lambda^2}{a^2}-1>0, \quad \text{if} \quad \frac{\Lambda^2}{a^2}>e,$$

i.e. $f(a^2)$ will be a monotonically increasing function of a^2 for large enough Λ^2. Hence, if $\Lambda^2/\sigma_2^2>\Lambda^2/(\sigma_2^2+\Delta_2^2)>e$, we will have the inequality

$$\sigma_2^2\ln(\Lambda^2/\sigma_2^2)<(\sigma_2^2+\Delta_2^2)\ln[\Lambda^2/(\sigma_2^2+\Delta_2^2)]. \quad (25)$$

Applying Eq.(25) to Eqs.(23) and (24), we are led to the condition

$$G_S/H_S<2/3. \quad (26)$$

On the other hand, by Eqs.(23) and (24) we have the inequality

$$1/2G_S-3\Lambda^2/2\pi^2<1/2H_S-\Lambda^2/\pi^2$$

which may be changed into

$$G_S/H_S>1/(1+H_S\Lambda^2/\pi^2). \quad (27)$$

Hence only if Eqs.(26) and (27) are satisfied, we can have a non-zero solution (σ_2,Δ_2) as a minimum point of $V(\sigma,|\Delta|)$. It may be seen by comparing Eq.(21) with Eq.(27) that the minimum points $(0,\Delta_1)$ and (σ_2,Δ_2) can not coexist, so in each case the only minimum point (for $\sigma>0$) will correspond to the ground state of the system, especially the solution (σ_2,Δ_2) implies coexistence of the quark-antiquark and diquark condensates in the ground state.

All the above results can be summarized as follows. Locations of the minimum points of the effective potential $V(\sigma,|\Delta|)$ depend on relative values of G_S and H_S. The minimum points will respectively be at

$$(\sigma,|\Delta|)=\begin{cases}(0, & \Delta_1)\\ (\sigma_2, & \Delta_2)\\ (\sigma_1, & 0)\end{cases} \quad \text{if} \quad \begin{cases}0 & \leq & G_S/H_S<1/(1+H_S\Lambda^2/\pi^2)\\ 1/(1+H_S\Lambda^2/\pi^2) & < & G_S/H_S<2/3\\ & & G_S/H_S>2/3\end{cases}, \quad (28)$$

where σ_1 and Δ_1 are determined by Eqs.(19) and (20) separately, and (σ_2,Δ_2) by Eqs.(23) and (24). Combining Eq.(28) with Eqs.(15) and (16), we have shown in Fig.1 the corresponding regions to the three different phases in the coupling constants G_S-H_S plane.

(location of Fig.1)

It may be seen that in the region with non-zero condensates, the pure quark-antiquark condensate phase and the pure diquark condensate phase is separated by a coexistence phase of the two condensates. in each phase, either (0,Δ_1) or (σ_2,Δ_2) or (σ_1,0) is the only minimum point of $V(\sigma,|\Delta|)$ and no other minimum point arises, so there is no metastable state in the model. We indicate that the above $G_S - H_S$ phase diagram in the vacuum of a NJL model is first obtained.

The results show that the pure quark-antiquark condensates phase and the phase with diquark condensates is distinguished merely by the ratio $G_S/H_S = 2/3$ and the diquark condensates could emerge only if $H_S\Lambda^2/\pi^2 > 1/2$ and G_S/H_S is below the critical value 2/3. From the expressions (11) and (12) of the effective potential $V(\sigma,|\Delta|)$ and the follow-up discussions it is not difficult to see that the critical value 2/3 reflects the following fact: in the two-flavor NJL model only the two colors (red and green) of quarks anticipate in the diquark condensates but all the three colors (red, green and blue) of quarks get into the quark-antiquark condensates.

IV. CONCLUSIONS AND DISCUSSIONS

We have analyzed the effective potential of a two-flavor NJL model at $T = \mu = 0$ which contains two order parameters σ and $|\Delta|$ separately corresponding to quark-antiquark and diquark condensates and proven that the mutual competition between σ and $|\Delta|$ is decided by the ratio of the scalar quark-antiquark and the scalar diquark four-fermion coupling constant G_S and H_S. The results indicate that even in vacuum, the diquark condensates could either exist alone or coexist with quark-antiquark condensates when $G_S/H_S < 2/3$ and H_S is large enough. However, only the pure quark-antiquark condensates could be generated when $G_S/H_S > 2/3$. The last conclusion coincides with the result obtained in Ref. [14] based on a random matrix model, where the critical value of G_S/H_S above which no diquark condensates arise is $2/N_c$ for a color $SU_c(N_c)$ group. In fact, generalization of the analysis in present paper to $SU_c(N)$ case is direct. In doing so, not only the critical value $2/N_c$ will be derived but also a complete $G_S - H_S$ phase diagram similar to Fig.1 can be obtained. The conclusion that a very large coupling constant H_S in an NJL model could lead to diquark condensates in vacuum is also touched on in Ref.[15], but no further details were given there. It is also indicated that, in the above both models no corresponding $G_S - H_S$ phase diagram was given.

The above conclusions are reached in the chiral limit. If a non-zero degenerate bare quark mass $m_u = m_d = m$ is included, then the effective potential will have a little more complicated expression thus a complete analytic demonstration will become more difficult. However, the essential part of the discussions including determination of the conditions of emergence of the minimum points $(\sigma_1, 0)$ and (σ_2, Δ_2) can still be conducted analytically and the relevant essential conclusions reached in the $m = 0$ case will keep unchanged. In particular, we can definitely give the critical value of G_S/H_S at which the pure quark-antiquark condensate phase will transfer to the coexistence phase of quark-antiquark and diquark condensates, but now the value will change from 2/3 into $2\sigma_{min}/3(\sigma_{min} + m)$, where σ_{min} is the value of the order-parameter related to the quark-antiquark condensates in the ground state. Because the extra factor $\sigma_{min}/(\sigma_{min} + m) \leq 1$, so in nor-zero bare quark mass case, appearance of the diquark condensates requires stronger diquark interactions than the ones in the chiral limit.

In our discussions, the coupling constants G_S and H_S have been viewing as independent and changeable parameters, this is only a theoretical assumption for the purpose of exclusively researching mutual competition between the diquark and the quark-antiquark condensates in the model. In fact, G_S and H_S are interrelated via the Fierz transformations. Therefore, for a given NJL model in advance, the ratio G_S/H_S in Eq.(28) must be understood as its derived value in the Hartree approximation after the Fierz transformations are implemented.

By the above results, it seems that, theoretically, a general two-flavor NJL model does not remove existence of diquark condensates in vacuum, as long as the diquark interactions are strong enough. However, this can not be realistic case. Since a NJL model is usually used as a low energy phenomenological description of QCD, its coupling forms and strengths must be restricted by the underlying QCD theory and/or phenomenology, hence the ratio G_S/H_S can not be arbitrary. If the underlying Lagrangian or all the couplings of $(\bar{q}q)^2$-form are known, then via the Fierz transformations, the effective coupling G_S including direct and exchange interactions and the coupling H_S in the diquark channel can be fixed uniquely in the Hartree approximation, and so is the ratio G_S/H_S. However, when the underlying Lagrangian is unknown, one can write down only the partial coupling terms of $(\bar{q}q)^2$-form by symmetries and/or phenomenology, then the effective G_S and H_S derived by the Fierz transformations (still in the Hartree approximation) will not be unique, since they will change with different selections of the undetermined coupling terms of $(\bar{q}q)^2$-form.

However, our theoretical results (28), combined with phenomenological fact that there is no diquark condensates in the vacuum of QCD, will be able to place a useful restriction to the above different selections. In fact, for any NJL model which is intended to simulate QCD, whatever initial coupling terms as a starting point of the model building are selected, via the Fierz transformations, the derived least value of the effective ratio G_S/H_S of the positive scalar quark-antiquark and the positive scalar diquark coupling constants in the Hartree approximation must be larger than 2/3. This restriction is applicable to any realistic NJL model.

Two examples of the models which obey the above restriction are the models where the four-fermion interactions are assumedly induced by heavy gluon exchange or by instantons in QCD. These interactions can considered as some underlying more microscopic ones. In both models, the resulting ratio G_S/H_S via the Fierz transformations are equal to 4/3 [12] which is obviously larger than the critical value 2/3 for generation of the diquark condensates. So in these models we will have only the quark-antiquark condensates surviving. In Ref.[13] a slightly different conclusion was reached in an instanton-induced model with fixed values of G_S and H_S. There for $N_c = 3$ case, in the ground state similarly only the quark-antiquark condensates exist, but a pure diquark condensates also exist in a metastable state. However, it is noted that in that model the corresponding ratio G_S/H_S was not taken to be the conventional value 4/3 and a further relation between the coupling constants and the instanton density was used.

Another more phenomenological model is the conventional chiral $SU_{fL}(2) \otimes SU_{fR}(2)$-invariant four-fermion model with interactions [4]

$$\mathcal{L}_{int} = g[(\bar{q}q)^2 + (\bar{q}i\gamma_5\vec{\tau}q)^2].$$

Via the Fierz transformations we may obtain the effective Lagrangian

$$\mathcal{L}_{int}^{eff} = \mathcal{L}_{q\bar{q}} + \mathcal{L}_{qq}$$

with

$$\begin{aligned}\mathcal{L}_{q\bar{q}} &= G_S(\bar{q}q)^2 + G_P(\bar{q}i\gamma_5 q)^2 + G_P^\tau(\bar{q}i\gamma_5\vec{\tau}q)^2 + \cdots,\\ \mathcal{L}_{qq} &= H_S(\bar{q}i\gamma_5\tau_2\lambda_A q^C)(\bar{q}^C i\gamma_5\tau_2\lambda_A q)\\ &\quad + H_P(\bar{q}\tau_2\lambda_A q^C)(\bar{q}^C\tau_2\lambda_A q) + \cdots,\end{aligned}$$

where the ellipses indicate the possible vector, axial-vector and tensor coupling terms. It is emphasized that the effective Lagrangian $\mathcal{L}_{int}^{eff}$ is used only in the Hartree approximation [12]. For the two-flavor and three-color case, we get that the corresponding effective coupling constants

$$G_S = G_P^\tau = 11g/12,\ \ G_P = g/12,\ \ H_S = -H_P = g/4$$

Obviously, owing to that $G_S/H_S = 11/3 > 2/3$, it is impossible to generate the scalar diquark condensates in this model. The maximal attractive channels are still the original starting point, i.e. the coupling terms $(\bar{q}q)^2$ and $(\bar{q}i\gamma_5\vec{\tau}q)^2$. The possible condensates will be $\langle\bar{q}q\rangle$ (and/or the π condensates $\langle\bar{q}i\gamma_5\tau_a q\rangle$) and they are related to spontaneous chiral symmetry breaking only.

In brief, in a few known possibly realistic QCD-analogous two-flavor NJL models, owing to that the condition to generate the diquark condensates in vacuum is not satisfied, there could be only the quark-antiquark condensates in ground states of these models at $T = \mu = 0$. These models are merely suitable for description of chiral symmetry breaking and have nothing to do with color superconductivity at $T = \mu = 0$. This certainly reflects the reality of QCD in vacuum.

[1] Y. Nambu, Phys. Rev. Lett. **4** (1960) 380.
[2] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122** (1961) 345.
[3] J. Goldstone, Nuovo Cimento **19** (1961) 154 .
[4] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **124** (1961) 246 .
[5] D. Weingarten, Phys. Rev. Lett. **51** (1983) 1830; S. Nussinov, Phys. Rev. Lett. **51** (1983) 2081; E. Witten, Phys. Rev. Lett. **51** (1983) 2351; C. Vafa and E. Witten, Nucl. Phys. B**234** (1984) 173.
[6] B. R. Zhou, Phys. Lett. B**215** (1988) 364; B**235** (1990) 412(E); Commun. Theor. Phys.**15** (1991) 319; **18** (1992) 347; **19** (1993) 377 .
[7] V. A. Miransky, *Dynamical symmetry breaking in Quantum Field Theory*, (World Scientific Publishing Co.,Singapore, 1993) and the references therein.
[8] J. C. Collins and M. J. Perry, Phys. Rev. Lett. **34** (1975) 1353.
[9] M. Alford, K. Rajagopal, and F. Wilczek, Phys. Lett. B**422** (1998) 427; Nucl Phys. B**537** (1999) 443.
[10] R. Rapp, T. Schäfer, E. V. Shuryak, and M.Velkovsky, Phys. Rev. Lett. **81** (1998) 53.
[11] J. Berges and K. Rajagopal, Nucl. Phys. B**538** (1999) 215 .
[12] M. Buballa, Phys. Rep. **407** (2005) 205.
[13] D. Diakonov, H. Forkel, and M. Lutz, Phys. Lett.B **373** (1996) 147.
[14] B. Vanderheyden and A.D. Jackson, Phys. Rev. D**61** (2000) 076004.
[15] S. B. Rüster and D. H. Rischke, Phys. Rev. D**69** (2004) 045011.
[16] J.Jaeckel and C. Wetterich, Phys. Rev. D, **68** (2003) 025020.
[17] M. Asakawa and K. Yazaki, Nucl. Phys. A**504** (1989) 668.
[18] Y. Nambu, Phys. ReV. **117** (1960) 648 ; L. P. Gorkov, JETP **7** (1958) 993.
[19] D. Ebert and H. Reinhardt, Nucl. Phys. B**271** (1986) 188; B. Rosenstein, B. J. Warr, and S. H. Park, Phys. Rep. **205** (1991) 59; Zhou Bang-Rong, Commun. Theor. Phys. **39** (2003) 663.

FIGURE

Fig.1 The corresponding regions to the three different phases (0, Δ_1), (σ_2, Δ_2) and (σ_1,0) in $G_S - H_S$ plane (in dimensionless couplings $y = G_S\Lambda^2/\pi^2$ and $x = H_S\Lambda^2/\pi^2$).

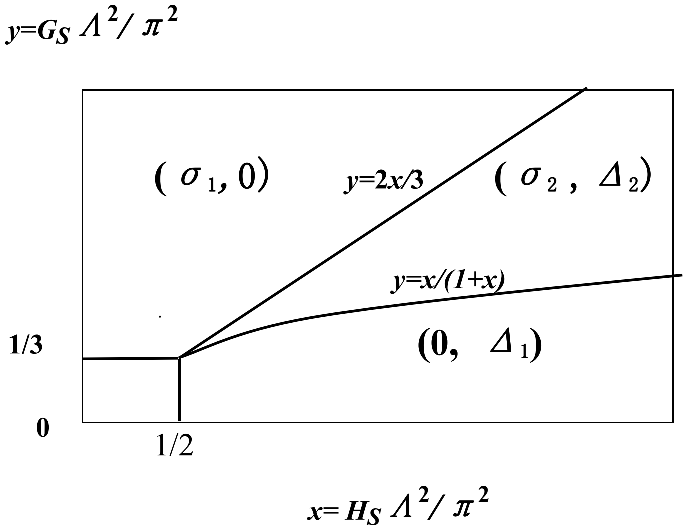

Fig.1